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Partial translations of semigroups

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To Jeanette Ryan with deep affection

1. Introduction. If S is semigroup, an element ϱ of $\mathscr{P}_S$ will be called a *partial right translation* of S if whenever $a\varrho$, $a\in S$, is defined then $(xa)\varrho$ is defined for all $x\in S$ and $x(a\varrho)=(xa)\varrho$. This definition was introduced in [6] to extend that of "one-to-one partial right translation" given by CLIFFORD and PRESTON ([1], vol. 1, page 32). Clearly both the domain of ϱ and the range of ϱ are left ideals of S and so the notion of partial right translation coincides with that of "left S-translation" discussed by STEINFELD [4]. However we prefer the former terminology since the concept of partial right translation generalises in a natural way that of "right translation".

Our first aim in this paper will be to describe a class of semigroups embeddable in inverse semigroups and thus provide an alternative account of some work done by SCHEIN [2]. After this we investigate the problem of determining conditions under which a semigroup S can be embedded in a semigroup T such that every partial right translation of S is induced by a right translation of T.

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2. Replete semigroups. All terminology will be that of [1]. Thus $\Lambda(S)$ $[P(S)]$ denotes the semigroup of all left [right] translations of S and $\Lambda_0(S)$ $[P_0(S)]$ the semigroup of all inner left [right] translations of S. We let $\mathscr{L}(S)$ $[\mathscr{R}(S)]$ denote the set of all partial left [right] translations of S.

Suppose $\varrho_1, \varrho_2\in\mathscr{R}(S)$ and $\varrho_1\varrho_2\neq\Box$ (that is, ran $\varrho_1\cap$dom $\varrho_2\neq\Box$) and put

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$L = (\operatorname{ran} \varrho_1 \cap \operatorname{dom} \varrho_2)\varrho_1^{-1}$. Then $\operatorname{dom} \varrho_1\varrho_2 = L$ and if $a \in L$ and $x \in S$, then $a \in \operatorname{dom} \varrho_1$, $(xa)\varrho_1 \in \operatorname{dom} \varrho_2$, and we have

$$(xa)\varrho_1\varrho_2 = ((xa)\varrho_1)\varrho_2 = (x(a\varrho_1))\varrho_2 = x(a\varrho_1)\varrho_2 = x(a\varrho_1\varrho_2).$$

Since therefore the only way $\mathscr{L}(S)$ $[\mathscr{R}(S)]$ can fail to be a semigroup under composition is that some product of two elements equals $\Box$, we shall on occasion regard $\Box$ as a partial left [right] translation of S: this accords with the standard terminology for $\mathscr{I}_X$. We note in passing that if $S = S^0$, then $0 \in L$ for every left ideal of S and so $\varrho_1\varrho_2 \neq \Box$ for all $\varrho_1, \varrho_2 \in \mathscr{R}(S)$; thus, when S contains a zero, $\mathscr{L}(S)$ $[\mathscr{R}(S)]$ is a semigroup properly containing $\Lambda(S)$ $[P(S)]$ (since the identity on $\{0\}$ is a partial left (right] translation of S).

If L is a left ideal of S and $a \in S$, we call $\varrho \in \mathscr{P}_S$ such that $\operatorname{dom} \varrho = L$ and $x\varrho = xa$ for all $x \in L$ the *partial inner right translation* (briefly, "pirt") of L *induced by* a and let $\mathscr{R}_0(S)$ denote the set of all such pirts. With the above convention $\mathscr{R}_0(S)$ is a subsemigroup of $\mathscr{R}(S)$ containing $P_0(S)$. The notions of partial inner left translation ("pilt" for short) and of $\mathscr{L}_0(S)$ are defined dually.

As usual (see [1], vol. 2), a non-empty subset A of a semigroup S is called *left unitary* if $xy \in A$ implies $y \in A$. A non-trivial example of a semigroup in which every left ideal is left unitary is provided by Exercise 6, § 1.11 [1]. The following result is analogous to those of Exercises 4 and 6 of § 1.3 [1]; the proof is straightforward and so is omitted.

Proposition 1. *If S is a semigroup then*
(i) *if each left ideal of S contains a right identity then $\mathscr{R}(S) = \mathscr{R}_0(S)$;*
(ii) *if $S = S^2$ then every $\varrho \in \mathscr{R}(S)$ with left unitary domain commutes with every $\lambda \in \mathscr{L}(S)$ with right unitary domain.*

STEINFELD [3] has used the notion of a one-to-one partial right translation to characterise completely 0-simple semigroups as those semigroups $S = S^0$ having the form $S = \bigcup\{Se_i : i \in I\}$ where $e_i^2 = e_i$ and Se_i are 0-minimal left ideals such that for each $i, j \in I$, there exists a one-to-one partial right translation from Se_i onto Se_j. In [4] he omits the assumption of 0-minimality to consider a wider class of semigroups, the so-called similarly decomposable semigroups, and extends the Rees theory for completely 0-simple semigroups to certain matrix semigroups over semigroups with a zero and identity (for a survey of this area see [5]). In doing this he proves the following interesting results; we shall call two left ideals L_1, L_2 *right equivalent* if there is a one-to-one partial right translation from L_1 onto L_2.

Proposition 2. *If $S = S^0$ and e_1, e_2 are non-zero idempotents in S then the following are equivalent:*

(i) Se_1 *and* Se_2 *are right equivalent,*
(ii) *there exist* $u, v \in S$ *such that* $uv = e_1$ *and* $vu = e_2$,
(iii) $e_1 S$ *and* $e_2 S$ *are left equivalent.*

Proposition 3. *If* $S = S^0$ *and* e_1, e_2 *are non-zero idempotents in* S *and* Se_1, Se_2 *are right equivalent, then the semigroups* $e_1 Se_1$ *and* $e_2 Se_2$ *are isomorphic.*

Proposition 4. *If* $S = S^0$ *and* Se_1, Se_2 *are* 0*-minimal left ideals of* S *then either* Se_1 *and* Se_2 *are right equivalent or the only partial inner right translation of* Se_1 *induced by any* $a \in Se_2$ *is the zero translation of* Se_1.

In this section we use the notion of a one-to-one partial right translation to provide an alternative account of Schein's characterisation of those semigroups with identity that are embeddable in inverse semigroups ([3]; see also [1], § 11.4).

Let S be a semigroup satisfying:

(1) for each $a \in S$, there exists an idempotent $e \in S$ such that $ea = a$ and if $xa = ya$ then $xe = ye$.

If $S = S^0$, write $e_0 = 0$ and more generally for each non-zero $a \in S$, choose and fix an idempotent, denoted by e_a, satisfying (1) and put $H(S) = \{e_a : a \in S\}$. We call S a *replete* semigroup if S satisfies (1) and $H(S)$ can be chosen to satisfy the further properties: (2) $e_a e_b = e_b a_a$ and (3) $ae_b = e_{ab} a$.

Before proceeding we note that if S is replete then for $a, b \in S$, $e_a ab = e_{ab} ab$ implies that $e_a e_{ab} = e_{ab}$, and this latter identity will be used without further mention.

A result similar to the following appeared in [6]; its proof is motivated by that of Theorem 1.20 [1].

Theorem 1. *Every inverse semigroup is replete and every replete semigroup is embeddable in an inverse semigroup.*

Proof. If S is inverse and $a \in S$, put $e_a = aa^{-1}$ and $H(S) = \{e_a : a \in S\}$. It is easy to check (using Theorem 1.17 and Lemma 1.18 of [1]) that S is replete with respect to $H(S)$.

Suppose S is replete and for each $a \in S$, put $a\varphi = \varrho_a | Se_a$; note that if $S = S^0$ then $0\varphi = \varrho_0 | \{0\}$. We assert that φ embeds S into $\mathscr{I}_S$. For, if $x, y \in Se_a$ and $x\varrho_a = y\varrho_a$, then using (1), $x = xe_a = ye_a = y$. Hence $a\varphi \in \mathscr{I}_S$ for all $a \in S$. If $a\varphi = b\varphi$ then $Se_a = Se_b$ and so using (2) we have $e_a = e_b$. Hence

$$a = e_a a = e_a \varrho_a = e_a \varrho_b = e_a b = e_b b = b$$

and so φ is one-to-one. Now suppose $ab \neq 0$ and note that $x \in \operatorname{dom}(a\varphi b\varphi)$ if and only if $xe_a = x$ and $xae_b = xa$. Moreover these last two equations are together equivalent to $xe_{ab} = x$ which means $x \in \operatorname{dom}(ab)\varphi$. If $ab = 0$ then $ae_b b = 0 = 0b$

implies by (1) that $ae_b=0e_b=0$. Suppose $xe_a=x$ and $xae_b=xa$ for some non-zero $x\in S$. Then $xa=0=0a$ implies $x=xe_a=0e_a=0$, a contradiction. Hence $\operatorname{dom}(a\varphi b\varphi)=0$. In both cases therefore $\operatorname{dom}(a\varphi b\varphi)=\operatorname{dom}(ab)\varphi$ and so φ is a morphism.

3. Partial translations induced by total translations. It is readily observed that if $S=S^0$ and $\varrho\in\mathcal{R}(S)$ then $0\varrho=0$. In particular, if $\operatorname{dom}\varrho=\{0\}$ then ϱ can be regarded as the restriction of any $\sigma\in P(S)$ to $\{0\}$. It is therefore natural to ask whether every $\varrho\in\mathcal{R}(S)$ is induced by some $\bar\varrho\in P(S)$ in the sense that $\varrho=\bar\varrho|\operatorname{dom}\varrho$. This is certainly true for example when every left ideal of S equals Se for some idempotent $e\in S$. For then if $\varrho\in\mathcal{R}(S)$ has domain Se, we have for all $s\in S$

$$(se)\varrho = s(e\varrho) = s(e^2\varrho) = se(e\varrho) = (se)\varrho_a$$

where $a=e\varrho\in S$.

Example. Let S be the semigroup of all natural numbers under ordinary multiplication and let $\varrho\in\mathcal{P}_S$ have domain $2S$ and satisfy $(2n)\varrho=n$ for all $n\in S$. Then $m(2n)\varrho=mn=(m2n)\varrho$ for all $m\in S$ and so $\varrho\in\mathcal{R}(S)$. However if there exists $\bar\varrho\in P(S)$ such that $\varrho=\bar\varrho|\operatorname{dom}\varrho$, then $\bar\varrho=\varrho_a$ for some $a\in S$ (since $1\in S$) and so $n=(2n)\varrho=2na$ for all $n\in S$, a contradiction. Hence ϱ is not induced by any right translation of S.

In the light of this example, we now ask: can a semigroup S always be embedded in a semigroup T such that every $\varrho\in\mathcal{R}(S)$ is induced by some $\bar\varrho\in P(T)$? The next result introduces a sufficient condition under which an embedding can be achieved quite simply.

Theorem 2. *If $S=S^0$ is a semigroup in which every non-zero left ideal is left unitary, then every $\varrho\in\mathcal{R}(S)$ is induced by some $\bar\varrho\in P(S)$.*

Proof. For each $\varrho\in\mathcal{R}(S)$, we define $\bar\varrho\in\mathcal{I}_S$ by $x\bar\varrho=\begin{cases} x\varrho & \text{if } x\in\operatorname{dom}\varrho \\ 0 & \text{if } x\notin\operatorname{dom}\varrho.\end{cases}$ Then if $y\in\operatorname{dom}\varrho$ and $x\in S$, we have $x(y\bar\varrho)=x(y\varrho)=(xy)\varrho=(xy)\bar\varrho$ and if $y\notin\operatorname{dom}\varrho$, then $xy\notin\operatorname{dom}\varrho$, and we have $x(y\bar\varrho)=x0=0=(xy)\bar\varrho$. Finally, if $y\in S$ then $0(y\bar\varrho)=0=(0y)\bar\varrho$, and so we have shown that $\bar\varrho\in P(S)$ and by definition it induces $\varrho\in\mathcal{R}(S)$.

Clearly, if S is any semigroup in which every left ideal is left unitary (or equivalently, S is a disjoint union of minimal left ideals) then Theorem 2 can be applied to S^0, the semigroup S with a zero adjoined.

Finally we note that Tamura and Graham [7, Theorem 3] have determined the necessary and sufficient conditions on a semigroup S which ensure that any $\sigma\in P(S)$ is induced by an inner right translation of some semigroup T containing S as an ideal: we do not know if their result extends to the partial case.

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups.* **1** (1961) and **2** (1967), Amer. Math. Soc. (Providence, Rhode Island).

[2] B. M. Schein, Embedding semigroups in generalised groups, *Mat. Sb.* (N. S.), **55** (97) (1961), 379—400.

[3] O. Steinfeld, On semigroups which are unions of completely 0-simple subsemigroups, *Czechoslovak Math. J.*, **16** (191) (1966), 63—69.

[4] O. Steinfeld, On a generalization of completely 0-simple semigroups, *Acta Sci. Math.*, **28** (1967), 135—145.

[5] O. Steinfeld, On similarly decomposable semigroups, *to appear.*

[6] R. P. Sullivan, *A study in the theory of transformation semigroups,* Ph. D. Thesis, Monash University, 1969.

[7] T. Tamura and N. Graham, Certain embedding problems of semigroups, *Proc. Japan Acad.*, **40** (1964), 8—13.

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